A Quantitative Pólya-Szegő inequality

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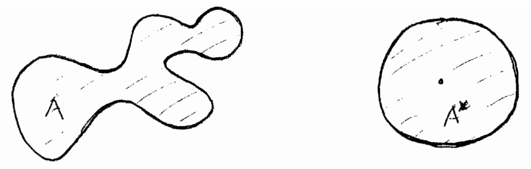
Outline

1. The Pólya-Szegő principle
   - Popular symmetrizations
   - The Pólya-Szegő inequalities
   - Toward the stability : the equality case

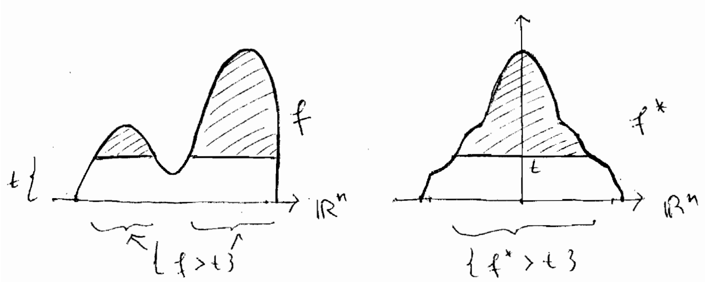
2. Stability result for the Steiner case
   - The main Theorem
   - Ideas from the proof

Schwarz Spherical Symmetrization about a point

$$A \subset \mathbb{R}^n \quad |A| < \infty \quad A^* := \{x \in \mathbb{R}^n \ : \ \omega_n |x|^n < |A|\}$$



$$f^*(z) := \inf\{t > 0 : |\{x \in \mathbb{R}^n : f(x) > t\}| \leq \omega_n |x|^n\}$$



drawing by Almut Burchard - 2009

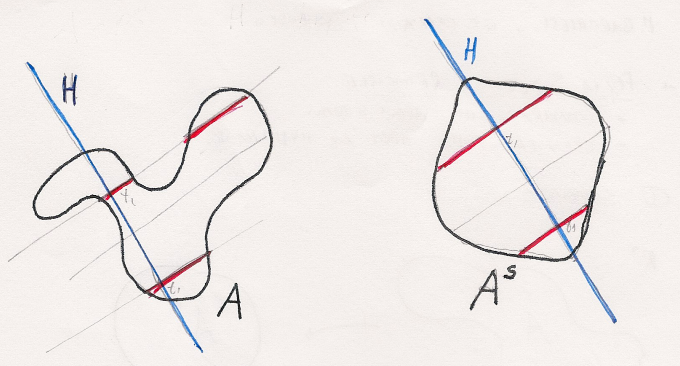
Steiner Symmetrization about an hyperplane

Given any measurable set $A \subset \mathbb{R}^n$, for every $x' \in \mathbb{R}^{n-1}$ we set

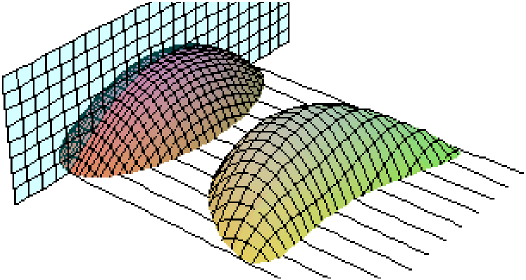
$$A_{x'} := \{x_n \in \mathbb{R} : (x', x_n) \in A\} \quad \text{and} \quad l(x') = \frac{1}{2}\mathcal{L}^1(A_{x'}).$$

Steiner symmetral A^s of A about the hyperplane $H := \{x_n = 0\}$

$$A^s := \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |x_n| < l(x')\}.$$



Steiner Symmetrization of a function



$u : \mathbb{R}^n \to [0, \infty)$ measurable such that for a.e. $x' \in \mathbb{R}^{n-1}$

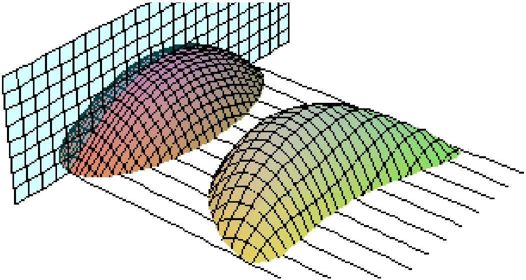
$$l_u(x', t) := \frac{1}{2}\mathcal{L}^1\left(\{x_n \in \mathbb{R} : u(x', x_n) > t\}\right) < \infty \quad \forall\, t > 0$$

The *Steiner rearrangement* of u with respect to $\{x_n = 0\}$ is

$$u^s(x) := \inf\left\{t > 0 : l_u(x', t) \le |x_n|\right\}.$$

graphic by Andrejs Treibergs - 2008

Steiner Symmetrization of a function



Remark (Segment property)

Denote by E_s the subgraph of u^s. E_s is a set enjoying the property that its intersection with any straight line L orthogonal to H is a segment, symmetric about H, whose length equals the (1-dimensional) measure of $L \cap E_u$

Pólya-Szegő inequalities

Theorem

If $u \in W_0^{1,p}(D)$ *then* $u^s \in W_0^{1,p}(D^s)$ *and* $u^* \in W_0^{1,p}(D^*)$. *Moreover*

$$\int_{\mathbb{R}^n} |\nabla u|^p dz \geq \int_{\mathbb{R}^n} |\nabla u^s|^p dz$$

$$\int_{\mathbb{R}^n} |\nabla u|^p dz \geq \int_{\mathbb{R}^n} |\nabla u^*|^p dz.$$

Aim

The aim of our work is to provide a quantitative versions of the previous inequalities.

The main question

Pólya-Szegő deficit of u

$$D_p(u) := \int_{R^n} |\nabla u|^p \, dz - \int_{R^n} |\nabla u^s|^p \, dz$$

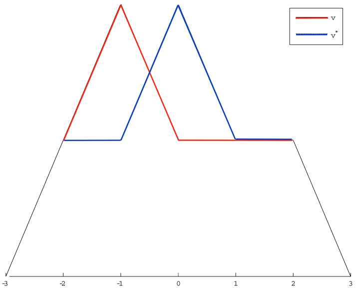
Question

Can the Pólya-Szegő deficit of an arbitrary $W_0^{1,p}$ function be used to estimate its asymmetry with respect to the hyperplane H, measured as a distance between u and u^s?

Remark

A relevant issue in connection with variational problems having Steiner symmetric extremal is that such an estimate should ensure that u is arbitrarily close to u^s when $D_p(u)$ is sufficiently small.

Counterexample - 1d - plateaus

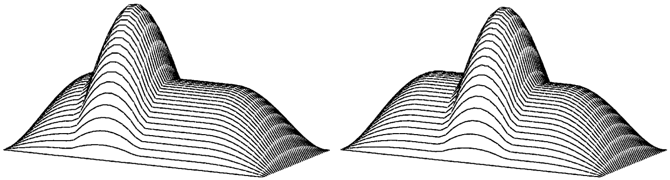


$$D_p(u) = 0 \not\Rightarrow \quad u = u^s$$

Main obstruction

The presence of a plateau below the top level in the graph of u

Counterexample - 2d - plateaus



$$D_p(u) = 0 \not\Rightarrow u = u^s$$

Problem : plateaus in the direction orthogonal to H

u is such that $\nabla_y u^s$ vanishes on a set of positive Lebesgue measure

$$\mathcal{L}^2\left(\left\{(x,y)\ :\ \nabla_y u^s = 0\,,\ u^s(x,y) < \sup_y u(x,y)\right\}\right) > 0$$

graphic by A. Cianchi and N. Fusco - 2006

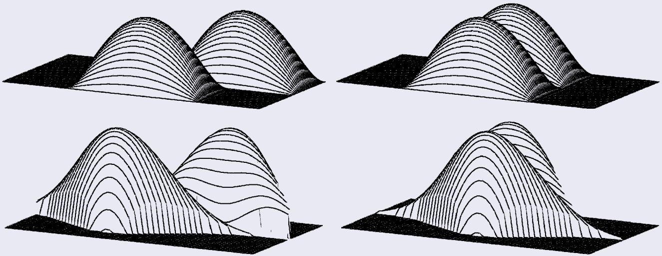
The Equality Case

Theorem (A. Cianchi & N. Fusco (2006))

Let $\Omega \subset \mathbb{R}^n$ be an open set satisfying some geometrical (essentially sharp) conditions and let u with no plateaus in the direction x_n, then

$$D_p(u) = 0 \quad \Rightarrow \quad u \equiv u^s \quad \textit{(up to translations)}$$

What a wrong Ω can cause



graphic by A. Cianchi and N. Fusco - 2006

The spherical symmetric case : a first guess

Remark (Counterpart of Brothers and Ziemer)

The previous result can be regarded as a Steiner symmetrization counterpart of Brothers and Ziemer's theorem on the spherical symmetry of minimal rearrangements, i.e. function with

$$\|\nabla u\|_p = \|\nabla u^*\|_p$$

Remark (Stability of Brothers and Ziemer)

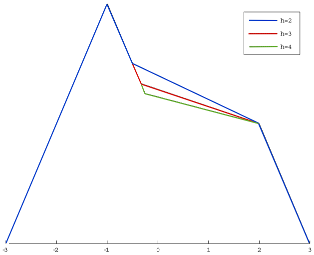
The result of Brothers and Ziemer is stable under perturbations of such an assumption, in the sense that the asymmetry of a minimal rearrangement can be estimated through $\mathcal{L}^n\left(\{\nabla u^* = 0\,,\ 0 < u^* < \text{ess-sup}\, u\}\right)$

First guess

Although the sole Pólya-Szegö deficit $D_p(u)$ of u is not sufficient to measure the distance of u from u^s, this should be possible if both $D_p(u)$ and $\mathcal{L}^n\left(\left\{(x,y)\ :\ \nabla_y u^s = 0\,,\ u^s(x,y) < \sup_y u(x,y)\right\}\right)$ are employed

Counterexample - almost plateaus

- $(v_h^*)' \neq 0, \; v_h' \neq 0 \; \mathcal{L}^1 - a.e.$
- $\lim_{h\to\infty} D_p(v_h) = 0 \; p \geq 1$
- $\|v_h - v_h^s\|_1 \geq const > 0$



Deduction

This counterexample, loosely speaking, shows that, when $D_p(u) > 0$, not only a large set where $\nabla_y u_s$ vanishes, but also a large set where $|\nabla_y u_s|$ is small, may allow u to be very asymmetric. Consequently, one can hope to control the asymmetry of a function u by means of $D_p(u)$ only if we have a control of the measure on the set where the $\nabla_y u_s$ is small

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Our choice of functions

When we can rule out "bad behavior"

- Concave functions
- α-concave functions ($\alpha \in (0, 1]$)
- Log-Concave functions with subgraph starshaped w.r.t. a ball
- Quasiconcave functions with a positive lower bound on $|\nabla u|$

Remark (They are not so bad classes)

The class of functions to which our stability results apply is large enough to include the solutions of the torsional rigidity problem and the first eigenfunction of the Laplacian operator with Dirichlet boundary conditions in smooth convex domains

Quantitative estimates

$$\Delta(u,u^s) := \int_{R^n} |\nabla u|^p \, dz - \int_{R^n} |\nabla u^s|^p \, dz$$

$$\lambda(u,u^s) := \inf_{h\in\mathbb{R}} \int_{\mathbb{R}^n} |u(x,y+h) - u^s(x,y)| \, dxdy$$

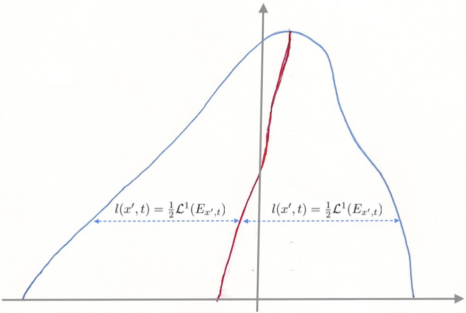
Theorem

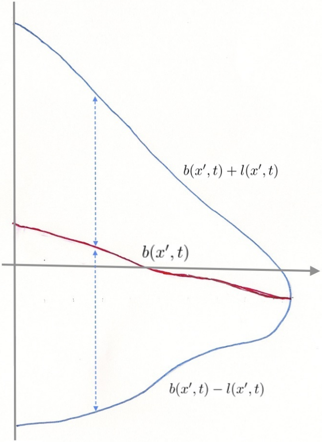
$D \subset \mathbb{R}^n$ bdd open convex set. Let $u \in W_0^{1,p}(D)$ be log-concave. Assume that the subgraph of u is star-shaped with respect to a ball of radius m. Then,

$$\lambda(u,u^s) \leq \begin{cases} c\,\dfrac{M^{n+2}}{m^{n+1}} |D|^{\frac{1}{p'}} \|\nabla u^s\|_{L^p}^{\frac{2-p}{2}} \Delta(u,u^s)^{\frac{1}{2}} & \text{if } 1 < p < 2; \\ c\,\dfrac{M^{n+2}}{m^{n+1}} |D|^{\frac{1}{p'}} \Delta(u,u^s)^{\frac{1}{p}} & \text{if } p \geq 2, \end{cases}$$

where $c = c(n,p)$ and $M = \max\{\|u\|_{L^\infty}, \textit{outer radius of } D\}$

Change of perspective : the functions l and b





- Ω : projection of E_u on $x_n = 0$
- $l = l(x', t)$: half measure of the 1-d sections
- $b = b(x', t)$ is the baricenter of the sections

The representation lemma

Lemma

- $D \subset \mathbb{R}^n$ *bounded open set*
- $u \in W^{1,p}(D)$ *($p \geq 1$) non-negative, continuous*
- E_u *has no plateaus and satisfies the segment property*

$$\Downarrow$$

$$\partial_t b + \partial_t l < 0 \text{ and } \partial_t b - \partial_t l > 0 \text{ a.e. in } \Omega$$

$$\int_D |\nabla u|^p \, dx = \int_\Omega \frac{(1 + |\nabla_{x'} b + \nabla_{x'} l|^2)^{\frac{p}{2}}}{|\partial_t b + \partial_t l|^{p-1}} \, dx' dt + \int_\Omega \frac{(1 + |\nabla_{x'} b - \nabla_{x'} l|^2)^{\frac{p}{2}}}{|\partial_t b - \partial_t l|^{p-1}} \, dx' dt$$

$$\int_D |\nabla u^s|^p \, dx = 2 \int_\Omega \frac{(1 + |\nabla_{x'} l|^2)^{\frac{p}{2}}}{|\partial_t l|^{p-1}} \, dx' dt$$

How to deal with the deficit

$$\Delta(u,u^s) = \int_\Omega \frac{(1+|\nabla_{x'}b+\nabla_{x'}l|^2)^{\frac{p}{2}}}{|\partial_t b+\partial_t l|^{p-1}} + \frac{(1+|\nabla_{x'}b-\nabla_{x'}l|^2)^{\frac{p}{2}}}{|\partial_t b-\partial_t l|^{p-1}} - 2\frac{(1+|\nabla_{x'}l|^2)^{\frac{p}{2}}}{|\partial_t l|^{p-1}} dx'dt$$

Careful estimates on the second order increment of the function

$$f_p : \mathbb{R}^{n-1}\times(0,\infty)\to\mathbb{R} \quad f_p(x) := \frac{(1+|x'|^2)^{\frac{p}{2}}}{x_n^{p-1}}$$

$$\Downarrow$$

$$\underbrace{\int_\Omega \frac{|\nabla_{x'}b|}{\sqrt{1+|\nabla_{x'}l|^2}}\,dx'dt}_{I} + \underbrace{\int_\Omega \frac{|\partial_t b|}{|\partial_t l|}\,dx'dt}_{II} \le \begin{cases} c\mathcal{L}^n(D)^{\frac{1}{p'}}\,\|\nabla u\|_{L^p}^{\frac{2-p}{2}}\,\Delta(u,u^s)^{\frac{1}{2}} & p<2 \\ c\mathcal{L}^n(D)^{\frac{1}{p'}}\,\Delta(u,u^s)^{\frac{1}{p}} & p\ge 2 \end{cases}$$

How to deal with the deficit

$$\Delta(u,u^s) = \int_\Omega \frac{(1+|\nabla_{x'} b + \nabla_{x'} l|^2)^{\frac{p}{2}}}{|\partial_t b + \partial_t l|^{p-1}} + \frac{(1+|\nabla_{x'} b - \nabla_{x'} l|^2)^{\frac{p}{2}}}{|\partial_t b - \partial_t l|^{p-1}} - 2\frac{(1+|\nabla_{x'} l|^2)^{\frac{p}{2}}}{|\partial_t l|^{p-1}} dx'dt$$

Careful estimates on the second order increment of the function

$$f_p : \mathbb{R}^{n-1} \times (0,\infty) \to \mathbb{R} \quad f_p(x) := \frac{(1+|x'|^2)^{\frac{p}{2}}}{x_n^{p-1}}$$

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Aim

We want to get rid of the dependence on l in I and II, since the L^1 distance between u and u^s can be estimated by the L^1 norm of the baricenter b

Ingredients of the estimates

1. Use *convexity of level sets* to estimate

$$\frac{1}{\sqrt{1+|\nabla_{x'} l(x',t)|^2}} \geq \frac{\mathrm{dist}(x', \partial\Omega_t)}{\sqrt{2}M} \quad \forall x' \in \Omega_t,$$

   where M is the maximum between $\|u\|_{L^\infty}$ and the outer radius of D

2. Use *log-concavity* to estimate

$$\frac{1}{|\partial_t l(x',t)|} \geq \frac{\ln 2}{M} \mathrm{dist}(t, \partial\Omega_{x'}).$$

3. Use *weighted Poincare* and *starshapedness* assump. to estimate

$$\begin{aligned}
\int_\Omega \frac{|\nabla_{x'} b|}{\sqrt{1+|\nabla_{x'} l|^2}} + \int_\Omega \frac{|\partial_t b|}{|\partial_t l|} &\geq \int_\Omega |\nabla b|\, \mathrm{dist}((x',t), \partial\Omega)\, dx'dt \\
&\geq c\left(\frac{m}{M}\right)^{n+1} \int_\Omega |b - b_0|\, dx'dt \\
&\geq c\left(\frac{m}{M}\right)^{n+1} \int_{R^n} |u(x', x_n + b_0) - u^s(x)| dx,
\end{aligned}$$

Thank you
for your attention!

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The Schwarz case for n-symmetric functions

Lemma

$D \subset \mathbb{R}^n$ *bounded open* n*-symmetric* $w \in W_0^{1,p}(D)$ *non-negative and* n*-symmetric*

$$\int_{\mathbb{R}^n} |w - w^*|\, dx \leq \begin{cases} c\, \mathcal{L}^n(D)^{\frac{1}{p'}+\frac{1}{n}} \|\nabla w^*\|_{L^p(\mathbb{R}^n)}^{\frac{2-p}{2}} \Delta(w,w^*)^{\frac{1}{2}} & \text{if } 1 < p < 2; \\ c\, \mathcal{L}^n(D)^{\frac{1}{p'}+\frac{1}{n}} \Delta(w,w^*)^{\frac{1}{p}} & \text{if } p \geq 2, \end{cases}$$

How to prove it

- Coarea formula
- Layer cake representation formula
- Quantitative isoperimetric inequality

The theorem for schwarz

$$\Delta(u,u^*) := \int_{R^n} |\nabla u|^p\, dz - \int_{R^n} |\nabla u^*|^p\, dz$$

$$\lambda(u,u^*) := \inf_{h\in\mathbb{R}} \int_{\mathbb{R}^n} |u(x,y+h) - u^*(x,y)|\, dxdy$$

Theorem

$D \subset \mathbb{R}^n$ bdd open convex set. Let $u \in W_0^{1,p}(D)$ be a non-negative and log-concave function. Assume that the subgraph of u is star-shaped with respect to a ball of radius m. Then

$$\lambda(u,u^*) \le \begin{cases} c\, \dfrac{M^{n+2}}{m^{n+1}} \mathcal{L}^n(D)^{\frac{1}{p'}} \|\nabla u^*\|_{L^p}^{\frac{2-p}{2}} \Delta(u,u^*)^{\frac{1}{2}} & \text{if } 1 < p < 2; \\ c\, \dfrac{M^{n+2}}{m^{n+1}} \mathcal{L}^n(D)^{\frac{1}{p'}} \Delta(u,u^*)^{\frac{1}{p}} & \text{if } p \ge 2, \end{cases} \qquad (1)$$

where $c = c(n,p)$ and $M = \max\{\|u\|_{L^\infty}, outer\ radius\ of\ D\}$

The theorem for schwarz

Theorem

Let $D \subset \mathbb{R}^n$ be a bounded open convex set and let $u \in W_0^{1,p}(D)$ be a non-negative and log-concave function. Assume that the subgraph of u is star-shaped with respect to a ball of radius m. Then

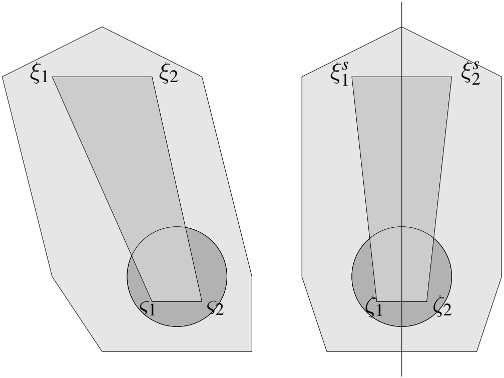
$$\lambda(u,u^*) \leq \begin{cases} c\,\dfrac{M^{n+2}}{m^{n+1}}\mathcal{L}^n(D)^{\frac{1}{p'}}\,\|\nabla u^*\|_{L^p}^{\frac{2-p}{2}}\,\Delta(u,u^*)^{\frac{1}{2}} & \text{if } 1<p<2; \\ c\,\dfrac{M^{n+2}}{m^{n+1}}\mathcal{L}^n(D)^{\frac{1}{p'}}\,\Delta(u,u^*)^{\frac{1}{p}} & \text{if } p \geq 2, \end{cases} \tag{2}$$

where $c = c(n,p)$ and $M = \max\{\|u\|_{L^\infty}, \textit{outer radius of } D\}$

Remark

The idea is to apply Steiner symmetrization n times along the n coordinate directions so to transform u in a n-symmetric function, and then to use the previous lemma

A comment about the proof



Remark

The property of the subgraph E_u of u of being star-shaped with respect to a ball of radius m is inherited by the subgraph E^s of u^s. More precisely, if E is star-shaped with respect to $B_m(\bar{\xi})$, for some $\bar{\xi} = (\bar{x}, \bar{t})$, then E^s is star-shaped with respect to $B_m((\bar{x}', 0, \bar{t}))$.

Thank you
for your attention!