First eigenpair asymptotics for singularly perturbed operastors

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Outline

We consider Dirichlet spectral problem in a bounded domain for a second order singularly perturbed elliptic operator with locally periodic coefficients. The talk will focus on the limit behaviour of the first eigenpair. Our goals are

- to describe the limit behaviour of the first eigenvalue and the logarithmic asymptotics of the first eigenfunction;
- to construct the second term of the asymptotics for a (perturbed) convection-diffusion operator;
- to address the problem of a choice of the solution to the limit problem which is responsible for the eigenfunction asymptotics, in the case of non-uniqueness.

In a smooth bounded domain $\Omega \subset \mathbb{R}^d$ we consider the following singularly perturbed elliptic operator:

$$\mathcal{L}_\varepsilon u = \varepsilon^2 a^{ij}(x, x/\varepsilon^\alpha)\frac{\partial^2 u}{\partial x_i \partial x_j} + \varepsilon b^j(x, x/\varepsilon^\alpha)\frac{\partial u}{\partial x_j} + c(x, x/\varepsilon^\alpha)u.$$

Here $\varepsilon > 0$ is a small positive parameters, $\alpha > 0$ is a given number. We assume that

- all the coefficients $a^{ij}(x, y)$, $b^i(x, y)$ and $c(x, y)$ are $[0, 1]^d$-periodic in y functions;
- the coefficient are smooth enough on $\overline{\Omega} \times \mathbb{T}^d$;
- the matrix $a(x, y)$ is uniformly elliptic, that is,

$$a(x, y)\xi \cdot \xi \geq \Lambda|\xi|^2$$

for some $\Lambda > 0$, all $\xi \in \mathbb{R}^d$ and all $(x, y) \in \overline{\Omega} \times \mathbb{T}^d$.

Spectral problem

We consider the Dirichlet spectral problem in Ω:

$$\mathcal{L}_\varepsilon u = \lambda u, \qquad u = 0 \quad \text{on } \partial\Omega \tag{1}$$

By the Krein-Rutman theorem, the principal eigenvalue λ_0^ε is real and simple, and the principal eigenfunction u_0^ε is positive under a proper normalization.

Our goal is to study the limit behaviour of the principal eigenpair (ground state), as $\varepsilon \to 0$.

Remarks

The first eigenpair (ground state) plays a crucial role when studying the large time behavior of solutions to the corresponding parabolic initial boundary problem. The first eigenvalue characterizes an exponential growth or decay of a typical solution, as $t \to \infty$, while the corresponding eigenfunction describes the limit profile of a normalized solution.

Also, since in a typical case the first eigenfunction shows a singular behavior, as $\varepsilon \to 0$, in many applications it is important to know the set of concentration points of u_ε, the so-called hot spots. This concentration set might consist of one point, or finite number of points, or a surface of positive codimension, or it might have more complicated structure.

Remarks

The asymptotic behaviour of the ground state depends crucially on whether $\alpha > 1$, or $\alpha < 1$, or $\alpha = 1$ (self-similar case).

Main difficulties

- The operator $\mathcal{L}_\varepsilon$ is non-selfadjoint;
- $\mathcal{L}_\varepsilon$ is singularly perturbed and at the same time has oscillating coefficients;
- Dirichlet boundary condition

Existing results

Boundary value problems for singularly perturbed elliptic operators have been widely studied in the existing literature.

- regular degeneration M.Vishik, L.Lusternik '57
- the Dirichlet problem for a convection-diffusion operator with a small diffusion and with a convection directed outward at the domain boundary was studied for the first time M.Freidlin and A.Wentzel '70. They used the large deviation techniques.
- The large deviation principle have also been used for studying the first eigenvalue of a second order elliptic operator being a singular perturbation of a first order operator, Yu. Kifer '80, '87.
- ground state asymptotics of a singularly perturbed elliptic operator on a compact Riemannian manifold, A.P. '98.

There are two main approaches to studying singularly perturbed elliptic operator:

- large deviation principle;
- viscosity solution techniques.

We are going to use viscosity solution approach. It was developed in the works of L.Evans, H.Ishii, P.Lions, B.Perthame, and others.

Our approach is also based essentially on homogenization results for first and second order nonlinear equations, P.Souganidis, P. Lions, L.Evans, and others.

WKB transformation

Since $u_\varepsilon > 0$ in Ω, we can represent u_ε as

$$u_\varepsilon(x) = e^{-W_\varepsilon(x)/\varepsilon}.$$

Then,

$$W_\varepsilon = -\varepsilon \log u_\varepsilon$$

We suppose that

$$\max_\Omega u_\varepsilon = 1$$

Then

$$W_\varepsilon \geq 0 \ \text{ in } \Omega, \qquad \min_{x\in\Omega} W_\varepsilon = 0.$$

Nonlinear equation

The function W_ε satisfies the following equation

$$-\varepsilon a^{ij}(x, x/\varepsilon^\alpha)\frac{\partial^2 W_\varepsilon}{\partial x_i \partial x_j} + H(\nabla W_\varepsilon, x, x/\varepsilon^\alpha) = \lambda_\varepsilon \qquad (2)$$

with the Hamiltonian

$$H(p, x, y) = a^{ij}(x, y)p_i p_j - b^j(x, y)p_j + c(x, y)$$

and the boundary condition

$$W_\varepsilon = +\infty \qquad \text{on } \partial\Omega.$$

Hamilton-Jacobi equation

Let $\overline{H}(p,x)$ be a function defined on $\mathbb{R}^d \times \overline{\Omega}$ such that

- $\overline{H}(p,x)$ is continuous;
- $\overline{H}(p,x)$ is convex in p;
- $\overline{H}(p,x) \geq m_1|p|^2 - C \quad$ for some $m_1 > 0$ and C.

Consider Hamilton-Jacobi equation

$$\overline{H}(\nabla W(x),x) = \lambda \qquad \text{in } \Omega. \tag{3}$$

with the so called state constraint boundary condition

$$\overline{H}(\nabla W(x),x) \geq \lambda \qquad \text{on } \partial\Omega. \tag{4}$$

Both the equation and the boundary condition are understood in viscosity sense.

Hamilton-Jacobi equation

Equivalently, we can rewrite (3)-(4) in the form

$$\overline{H}(\nabla W(x), x) \le \lambda \qquad \text{in } \Omega \tag{5}$$

$$\overline{H}(\nabla W(x), x) \ge \lambda \qquad \text{in } \overline{\Omega}, \tag{6}$$

Definition

A continuous function W is a solution to (5) if for any $x_0 \in \Omega$ and $C^2(\Omega)$ function ϕ such that $W - \phi$ attains a maximum at x_0, we have

$$\overline{H}(\nabla \phi(x_0), x_0) \le \lambda.$$

A continuous function W is a solution to (6) if for any $x_0 \in \overline{\Omega}$ and $C^2(\overline{\Omega})$ function ϕ such that $W - \phi$ attains a minimum at x_0, we have

$$\overline{H}(\nabla \phi(x_0), x_0) \ge \lambda.$$

Hamilton-Jacobi equation

Theorem

There exists a unique $\lambda \in \mathbb{R}$ such that problem (3)-(4) (or (5)-(6)) has a solution.

We denote this λ by $\lambda_{\overline{H}}$. Notice that a solution of problem (3)-(4) with $\lambda = \lambda_{\overline{H}}$ need not be unique.

Variational problem

Denote by $\overline{L}(\xi, x)$ the corresponding Lagrangian:

$$\overline{L}(\xi, x) = \max_p \{\xi \cdot p - \overline{H}(p, x)\}$$

Lemma

$$\lambda_{\overline{H}} = -\lim_{t\to\infty} \frac{1}{t} \inf \int_0^t L(\dot\eta(s), \eta(s))ds$$

where the infimum is taken over all absolutely continuous curves $\eta \,:\, [0, t] \longrightarrow \overline{\Omega}$.

Main results. The case $\alpha = 1$

Limit Hamiltonian

$\overline{H}(p,x)$ is the first eigenvalue (eigenvalue with the maximal real part) of the problem

$$a^{ij}(x,y)\frac{\partial^2\vartheta}{\partial y_i \partial y_j} + (b^j(x,y) - 2a^{ij}(x,y)p_i)\frac{\partial\vartheta}{\partial y_j}$$
$$+H(p,x,y)\vartheta = \overline{H}(p,x)\vartheta, \qquad \vartheta(y) \quad \text{is } Y\text{-periodic.} \tag{7}$$

Theorem

The eigenvalues λ_ε converge as $\varepsilon \to 0$ to the limit λ, which is the unique real number for which problem (3), (4) has a viscosity solution. The functions W_ε converge (along a subsequence) to a limit W uniformly on compacts in Ω, and every limit function W is a viscosity solution of (3), (4).

Limit Hamiltonian

For $\alpha > 1$ the limit Hamiltonian is defined by

$$\overline{H}(p,x) = \int_Y H(p,x,y)\vartheta(y)\,\mathrm{d}y \qquad (8)$$

where

$$H(p,x,y) = a^{ij}(x,y)p_i p_j - b^j(x,y)p_j + c(x,y),$$

and $\vartheta(y)$ is the unique $Y-$periodic solution of the equation

$$\frac{\partial^2}{\partial y_i \partial y_j}(a^{ij}(x,y)\vartheta) = 0$$

normalized by $\int_Y \vartheta(y)\,\mathrm{d}y = 1$.

Limit Hamiltonian

If $0 < \alpha < 1$, then the limit Hamiltonian $\overline{H}(p,x)$ is the unique number such that the problem

$$H(p + \nabla\vartheta(y), x, y) = \overline{H}(p,x) \tag{9}$$

has a $Y-$periodic viscosity solution $\vartheta(y)$; here $p \in \mathbb{R}^N$ and $x \in \overline{\Omega}$ are parameters.

Remarks

Remarks

For all $\alpha > 0$ we can show that $\overline{H}(p, x)$ satisfies the continuity, convexity and coerciveness conditions formulated above.

For $\alpha \geq 1$ the limit Hamiltonian is strictly convex in p.
For $\alpha < 1$ this might fail to hold, and the Hamiltonian need not be strictly convex.

The uniqueness of a limit point of W_ε is a very interesting issue.

Aubry set

$$\mathcal{A}_{\overline{H}} = \Big\{ y \in \overline{\Omega} \,:\, \sup_{\delta>0} \,\inf \Big(\int_0^t (\overline{L}(\dot{\eta}, \eta) + \lambda_{\overline{H}}) ds, \\ \eta(0) = \eta(t) = y,\ t > \delta \Big) = 0 \Big\}$$

From now on we consider a particular case

$$\mathcal{L}_\varepsilon u = \varepsilon^2 a^{ij}(x, x/\varepsilon)\frac{\partial^2 u}{\partial x_i \partial x_j} + \varepsilon b^j(x, x/\varepsilon)\frac{\partial u}{\partial x_j}.$$

Notice that $\alpha = 1$.

Effective drift

We define $\bar{b}(x)$ as follows:

$$\overline{b}^j(x) = \int_Y b^j(x,y)\theta^*(x,y)\mathrm{d}y,$$

where $\theta^*(x,y)$ is a periodic solution of

$$\frac{\partial^2}{\partial y_i \partial y_j}\Big(a^{ij}(x,y)\theta^*\Big) - \frac{\partial}{\partial y_j}\Big(b^j(x,y)\theta^*\Big) = 0$$

normalized by

$$\int_Y \theta^*(x,y)dy = 1;$$

x being a parameter.

Assumptions

We assume that

1. $\mathcal{A}_{\overline{H}}$ coincides with (a finite) set of hyperbolic fixed points of the ODE $\dot{x} = -\bar{b}(x)$;
2. $\mathcal{A}_{\overline{H}} \subset \Omega$.

One can show that

$$\frac{\partial \overline{H}}{\partial p_j}(0, x) = \bar{b}^j(x).$$

Under assumptions 1.–2.

$$\lambda_{\overline{H}} = 0,$$

and we want to construct the next term of the asymptotics of λ_ε. To this end we we denote the stationary points of the vector-field $b(x)$ by $\xi_1, \dots, \xi_N$. By our assumptions all these points belong to Ω, and their union coincides with the Aubry set of $\overline{H}$.

Define

$$B^{ij}(\xi_m) = \frac{\partial \bar{b}^j}{\partial x_i}(\xi_m), \qquad m = 1, 2, \dots, N.$$

We define $\sigma(\xi_m)$ to be the sum of the negative real parts of the eigenvalues of the matrix $-B^{ij}(\xi_m)$.

Theorem

Let $\alpha = 1$ and $c(x,y) = 0$. Then under assumptions 1.–2.,

$$\lambda_\varepsilon = \varepsilon\bar\sigma + o(\varepsilon)$$

with $\bar\sigma = \max\{\sigma(\xi_m)\ :\ m = 1,\dots,N\}$.

Denote

$$d_{\overline{H}-\lambda_{\overline{H}}}(x,y) = \inf\Big\{\int_0^t (\overline{L}(\dot\eta,\eta)+\lambda_{\overline{H}})\mathrm{d}\tau,\ \eta(0)=y,\eta(t)=x,t>0\Big\}.$$

Theorem

Let the maximum of $\sigma(\xi_m)$ be attained at exactly one point $\bar{\xi}$. Then

- $$W(x) = d_{\overline{H}}(x, \bar{\xi}), \qquad W(\bar{\xi}) = 0.$$
- $u_\varepsilon(\bar{\xi} + \sqrt{\varepsilon}z) \to u(z)$ *in* $C(K)$ *and weakly in* $H^1(K)$ *for every compact* K*, and the limit* u *is the unique positive eigenfunction of the Ornstain-Uhlenbeck operator,*

$$Q^{ij} \frac{\partial^2 u}{\partial z_i \partial z_j} + z_i B^{ij} \frac{\partial u}{\partial z_j} = \overline{\sigma} u \qquad \text{in } \mathbb{R}^N, \tag{10}$$

normalized by $u(0) = 1$*; here* $B^{ij} = B^{ij}(\bar{\xi})$ *and*

$$Q^{ij} = \frac{1}{2} \frac{\partial^2 \overline{H}}{\partial p_i \partial p_j}(0, \bar{\xi}).$$

Asymptotics in the presence of limit cycles

We proceed with studying operators of the form

$$\mathcal{L}_\varepsilon u = \varepsilon^2 a^{ij}(x, x/\varepsilon)\frac{\partial^2 u}{\partial x_i \partial x_j} + \varepsilon b^j(x, x/\varepsilon)\frac{\partial u}{\partial x_j} + c(x, x/\varepsilon)u,$$

and assume that the ODE

$$\dot{x} = \overline{b}(x)$$

has one or more hyperbolic limit cycles in Ω. It can also have some stationary points. Hyperbolicity means that the linearized Poincare map does not have eigenvalues on the unit circle.

Notice that this is again a particular case because the coefficient $c(x, x/\varepsilon)$ does not have a large parameter $1/\varepsilon$ in front of it.

We assume that the Aubry set $\mathcal{A}_{\overline{H}}$ of $\overline{H}$ in Ω consists of finite number of limit cycles and, probably, stationary points of the vector field $\overline{b}$.

If ξ_j is a zero of $\overline{b}$ (stationary point of the ODE) then we set

$\sigma_{1,j}(\xi)$ is the sum of negative real parts of the eigenvalues of the matrix

$$\Big(-\frac{\partial \overline{b}^i}{\partial x_j}(\xi_j)\Big)_{i,j=\overline{1,N}},$$

and

$$\sigma_{2,j}(\xi) = \int_Y c(\xi_j, y)\theta^*(\xi_j, y)\, dy.$$

Definition of σ_1 and σ_2

If ξ_j lies on a limit cycle then

$$\sigma_1(\xi_j) = \frac{1}{P} \sum_{|\Lambda_k(\xi_j)|<1} \log |\Lambda_k(\xi_j)|,$$

where $P > 0$ is the minimal period of the cycle,
and $\Lambda_k(\xi_j)$ are the eigenvalues of the linearized Poincaré map such that $|\Lambda_k(\xi_j)| < 1$;

$\sigma_2(\xi_j)$ is given by

$$\sigma_2(\xi_j) = \frac{1}{P}\int_0^P \int_Y c\big(x(t), y\big)\theta^*\big(x(t), y\big)\, \mathrm{d}y\, \mathrm{d}t$$

with $x(t)$ solving $\dot{x} = -\overline{b}(x),\ x(0) = \xi_j$.

The result

Theorem

Under the above assumptions on the coefficients we have

- $$\lim_{\varepsilon\to 0}\lambda_\varepsilon = \overline{\sigma} := \max_j\Big\{\sigma_1(\xi_j)+\sigma_2(\xi_j);\, \xi_j \in \mathcal{A}_{\overline{H}}\Big\}.$$
- *if the maximum in (6) is attained at exactly one connected component of the Aubry set (fixed point or limit cycle) then* W_ε *converge uniformly on compacts in* Ω *to the viscosity solution of*

$$\overline{H}(\nabla W, x) = \lambda_{\overline{H}} \quad \text{in } \Omega, \quad \overline{H}(\nabla W, x) \geq \lambda_{\overline{H}} \quad \text{on } \partial\Omega$$

vanishing on the aforementioned component of the Aubry set.