Trends in Non-Linear Analysis, IST, Lisboa 31/7-2/8/2014

Optimal design on thin domains and existence of optimal shapes

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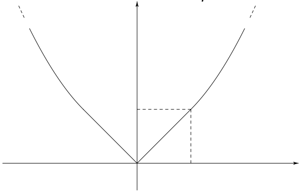
The problem

We consider the problem

$$m(s) := \inf \left\{ \int_D \varphi(\nabla u) \ : \ u \in H_0^1(D)\,, \ \int_D u = s \right\} \ ,$$

where $D \subset \mathbb{R}^2$ is a bounded simply connected domain, s a real parameter and

$$\varphi(y) := \begin{cases} \frac{1}{2}\left(1 + |y|^2\right) & |y| \geq 1 \\ |y| & |y| \leq 1 \end{cases}$$

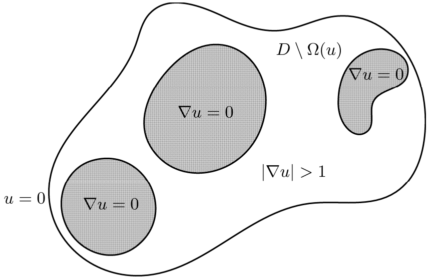


Does $m(s)$ admit a solution u such that

$$|\nabla u| \in \{0\} \cup \,]\,1, +\infty\,[\ \textit{ a.e. in } D \ ?$$

We call special solution such a minimizer for $m(s)$.

How it looks ?



$\Omega(u) := \{\nabla u = 0\}$ the plateau of u

$\Gamma(u) := \partial\Omega(u) \cap D$ the free boundary of u

Outline

1. Mechanical motivation: optimal design of thin torsion rods

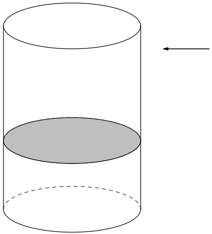
▷ GB, Fragalà, Seppecher, *Arch. Rat. Mech. Anal.* (2011).

▷ GB, Fragalà, Lucardesi, Seppecher, *SIAM J. Math. Anal.* (2012).

2. Optimality conditions, existence of a plateau and uniqueness.
3. Free boundary formulation and Cheeger sets
4. Existence results for special solutions
5. Further properties of special solutions and open problems

1. Optimal design of thin torsion rods

Minimize the compliance of an elastic material submitted to torsion, to be placed in a asymptotically thin design region with a prescribed volume fraction.



- shape optimization for the compliance [Allaire, Bonnetier, Cherkaev, Conca, Francfort, Gibiansky, Kohn, Strang, Jouve, Tartar]
- dimension reduction analysis [Acerbi, Braides, Buttazzo, Ciarlet, Fonseca, Le Dret, Mora, Muller, Murat, Raoult, Percivale, Tomarelli,Trabucho,Viano]

Compliance

The compliance of a linear elastic material placed in a subset $\Omega \subset \mathbb{R}^3$ submitted to an external load $F \in H^{-1}(\overline{\Omega}; \mathbb{R}^3)$, is the opposite of the energy at equilibrium. We associate the **shape** functional:

$$C(\Omega) := \sup \left\{ \langle F, u \rangle_{\mathbb{R}^3} - \int_{\Omega} j(e(u))\, dx \ : \ u \in C^\infty(\mathbb{R}^3; \mathbb{R}^3) \right\} .$$

- Since j is a quadratic: $C(\Omega) = \frac{1}{2}\langle F, \overline{u}\rangle$, with $\overline{u}$ optimal displacement.
- $j(z) = \frac{\lambda}{2}(\mathrm{tr}(z))^2 + \eta |z|^2$

The optimal design problem

For given load F and volume m of material, we have to solve the shape optmisation problem

$$\inf \left\{ C(\Omega) \ : \ |\Omega| = m \ , \ \Omega \subset Q, \right\}$$

As usual volume constraint is handled through Lagrange multiplier. For $k > 0$, we set

$$\phi(k) \ := \ \inf \left\{ C(\Omega) + k\,|\Omega| \ : \ \Omega \subseteq Q \right\}$$

Homogenization phenomena

Main features

- $\phi(k)$ is an ill-posed problem [Murat,Tartar].
- Minimizing sequences Ω_n tend to become more and more intricate fine mixture of voids and elastic material, i.e.
  $1_{\Omega_n} \to \theta$
  Here $\theta(x) \in [0,1]$ represents the local filling percentage of minimizing microstructures.

$$\theta(x) = \begin{cases} 0 & \text{no material} \\ 1 & \text{full material} \\ \in (0,1) & \text{fine mixture} \end{cases}$$

- Finding the variational problem solved by θ is challenging!

But.... a miracle happens for the 3D-1D reduction limit!

Under suitable assumptions on exterior load F^δ, it is possible to write explicitly the limit problem as $\delta \to 0$, for

$$\phi_\delta(k) := \inf\left\{ C(\Omega) + \frac{k}{\delta^2}|\Omega| \ : \ \Omega \subseteq Q_\delta \right\}$$

as a **convex well-posed problem** for densities $\theta \in L^\infty(\Omega; [0,1])$.

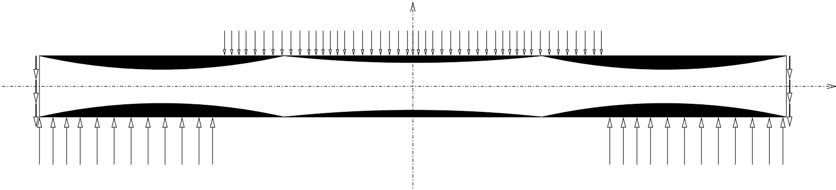
Question:

What can be said about optimal material distributions, namely about solutions $\overline{\theta}$ to the limit problem $\phi(k)$?

Do we have $\overline{\theta}$ with values into $\{0,1\}$ (true material), or into $[0,1]$ (composite material)?

Picture of the 3D-2D case:

For thin plates, classical solutions without homogenization regions always exist under the form of sandwich-like structures.



Optimal shape for a plate submitted to bending forces

[G.Bouchitté, I.Fragala, P.Seppecher *Arch. Rat. Mech. Anal.* 2011]

Back to 3D-1D: assumption on the load

- F is horizontal
- F has a Lebesgue negligible support
- $\langle F, u\rangle = \langle \Sigma, e(u)\rangle, \quad \Sigma \in L^2(Q; \mathbb{R}^{3\times 3}_{\rm sym})$ with $\Sigma_{33} = 0$

Examples: $\Omega = D \times I \ , \ I = (0,1)$

- $F = (\delta_1 - \delta_0)(x_3)(-\partial_2\psi(x'), \partial_1\psi(x'), 0) \qquad (\psi \in H^1_0(D))$
- $F = \rho(x_3)\tau_{\partial D}(x')\mathcal{H}^1 \llcorner \partial D \qquad (\rho \in L^2_m(I))$

Properties:

- $\langle F, u\rangle = 0 \qquad \forall\, u \in BN(Q) = \Big\{ e_{ij}(u) = 0 \ \forall (i,j) \neq (3,3) \Big\}$
- $\langle F, v\rangle = \langle m_F, c\rangle \quad \forall\, v \in TW(Q) = \Big\{ \big(c(x_3)(-x_2, x_1)\, ,\ v_3 \big) \Big\}$.
  $m_F := [[x_1 F_2 - x_2 F_1]]$ average momentum

The asymptotic analysis $\delta \longrightarrow 0$

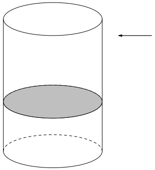
- **Small parameter problem:**

$$Q_\delta = \delta D \times I \quad , \quad F^\delta(x', x_3) = \delta^{-1} F(\delta^{-1} x', x_3)$$

$$\phi^\delta(k) := \inf_{A \subset Q^\delta} \left\{ \mathcal{C}^\delta(A) + k \frac{|A|}{\delta^2} \right\}$$

with $\quad \mathcal{C}^\delta(A) := \sup_{w \in H^1(Q^\delta; \mathbb{R}^3)} \{ \langle F^\delta, w \rangle - \int_A j(e(w)) \}$

$$\begin{cases} \delta \to 0 & \text{infinitesimal cross-section} \\ k = & \text{Lagrange multiplier} \\ (k \to +\infty & \text{for vanishing filling ratio}) \end{cases}$$



- **Reducing on fixed design** $Q = D \times I$ $(A \subset Q^\delta \rightsquigarrow \omega \subset Q)$

$$C^\delta(\omega) := \sup\left\{\delta^{-1}\langle F, u\rangle_{\mathbb{R}^3} - \int_\omega j(e^\delta(u))\,dx \;:\; u \in H^1(Q;\mathbb{R}^3)\right\}.$$

$$e^\delta(u) := \begin{bmatrix} \delta^{-2} e_{\alpha\beta}(u) & \delta^{-1} e_{\alpha 3}(u) \\ \delta^{-1} e_{\alpha 3}(u) & e_{33}(u) \end{bmatrix}.$$

As $\delta \to 0^+$, optimal displacements u_δ satisfy

$$\lim_\delta u_\delta = u \qquad \text{and} \qquad \lim_\delta \delta^{-1}\langle F, u_\delta\rangle = \langle F, v\rangle$$

for some $u \in BN(Q)$ and $v = \left(c(x_3)(-x_2, x_1),\ v_3\right) \in TW(Q)$.

- **Limit of ϕ_k^δ**

$$\delta \longrightarrow 0$$
$$+ \text{ relaxation}$$
$$\omega^\delta \subset Q \rightsquigarrow \theta \in L^\infty(Q;[0,1])$$

$$\phi(k) = \inf \left\{ \mathcal{C}^{lim}(\theta) + k \int_Q \theta \quad : \ \theta \in L^\infty(Q;[0,1]) \right\}$$

where

$$\theta = \text{local filling ratio of elastic material}$$

$$C^{lim}(\theta) \ := \ \sup_{c, v_3} \left\{ \langle m_F, c \rangle_{\mathbb{R}} - \kappa \int_Q \left| c'(x_3)(-x_2, x_1) + \nabla_{x'} v_3 \right|^2 \theta \, dx \right\}$$

Writing $\phi(k) \ = \ \inf_\theta \sup_{c,v} \ldots \ = \ \sup_{c,v} \inf_\theta \ldots$ we eliminate θ and are obtain (after dualizing with respect to pair (c, v)):

- **Dual problem on Q:**

$$\frac{\phi(k)}{2k} = \inf_{L^2(Q;\mathbb{R}^2)} \left\{ \int_Q \varphi(q) \ : \ \mathrm{div}_{x'} q = 0 \, , \ \int_D (x_1 q_2 - x_2 q_1) = -2\, M_F(x_3) \right\}$$

$(M_F(x_3) := \int_0^{x_3} m_F(s)\, ds)$

- **Localization on each section**
  - The dual form can be solved for $q(\cdot, x_3)$ section by section
  - The function $q(\cdot, x_3)$ is **divergence free on all** $\mathbb{R}^2$ and if $\mathbb{R}^2 \setminus D$ is connected

$$\exists u \in H_0^1(D) \ : \ q(\cdot, x_3) = (-\partial_2 u, \partial_1 u)$$

q optimal $\iff$ $q(\cdot, x_3) = (-\partial_2 u, \partial_1 u)$ where u optimal for $m(s)$ (with $s = M_F(x_3)$)

Link with *special* solutions?

Let u be a solution for

$$m(s) := \inf\left\{ \int_D \varphi(\nabla u) \ : \ u \in H^1_0(D)\,,\ \int_D u = s \right\}\ ,$$

and let θ solve $\phi(k)$ ($k = m'(s)$)

Then it holds (up to negligible subset)

$$\{0 < |\nabla u| < 1\} \ \subset \ \{0 < \theta < 1\} \ \subset \ \{0 < |\nabla u| \leq 1\}$$

Special solutions for $m(s)$ $\iff$ Classical solution for $\phi(k)$

(NO HOMOGENIZATION)

$$m(s) := \inf \left\{ \int_D \varphi(\nabla u) \ : \ u \in H_0^1(D) \, , \ \int_D u = s \right\} \ ,$$

Proposition

The map $s \mapsto m(s)$ *is convex even and* $\lim_{|s| \to \infty} \frac{m(s)}{s^2} = \tau_D > 0$

where (Saint-Venant torsional rigidity)

$$\tau_D := \frac{1}{2} \inf \left\{ \int_D |\nabla u|^2 \ : \ u \in H_0^1(D) \, , \ \int_D u = 1 \right\} \ ,$$

For every $s \in \mathbb{R}$*, the minimum* $m(s)$ *is achieved. Moreover*

If $m(s)$ *admits a special solution, then there is no other solution.*

Dual problem

The Fenchel conjugate of m reads

$$m^*(\lambda) = \min_{\sigma \in L^2(D;\mathbb{R}^2)} \left\{ \int_D \varphi^*(\sigma) \ : \ -\mathrm{div}\sigma = \lambda \right\} ,$$

where $$\varphi^*(\xi) = \frac{1}{2}\left(|\xi|^2 - 1\right)_+^2$$

Proposition (optimality conditions)

Let $s, \lambda \in \mathbb{R}$, $u \in H_0^1(D)$, and $\sigma \in L^2(D;\mathbb{R}^2)$. There holds the following equivalence

$$(i) \begin{cases} u \text{ solution to } m(s) \\ \sigma \text{ solution to } m^*(\lambda) \\ \lambda \in \partial m(s)\,. \end{cases} \iff (ii) \begin{cases} \int_D u = s \\ -\mathrm{div}\sigma = \lambda \\ \sigma \in \partial\varphi(\nabla u) \text{ a.e. in } D\,. \end{cases}$$

Remark: at every $s \neq 0$, $m(s)$ is differentiable and $m'(s) > 0$.

Argument for uniqueness

Take $\lambda \in \partial m(s)$ and a particular solution $\overline{\sigma}$ for $m^*(\lambda)$. Let

$$Q_s := \{|\overline{\sigma}| > 1\}$$

Then any solution u for $m(s)$ satisfies $\nabla u = \overline{\sigma}$ on $D \setminus Q_s$.
($\partial\varphi$ satisfies $\partial\varphi(\xi) = \xi$ if $|\xi| > 1$, and $\partial\varphi(0) = \overline{B(0,1)}$.)

$\square$

Existence of a plateau

Proposition

For every $s > 0$, any solution u to $m(s)$ is Lipschitz continuous and the maximal set $\{u = \max u\}$ has positive measure

Proof: Let $\lambda \in \partial m(s)$. Then, for every $v \in H_0^1(D)$:

$$\int_D \varphi(\nabla u) - \lambda \int_D u \;\leq\; \int_D \varphi(\nabla v) - \lambda \int_D v \ .$$

Take $t > 0$ and $v = \min\{u, t\}$. As $\varphi(z) \geq |z|$, we get

$$\int_{u>t} |\nabla u| \;\leq\; \lambda \int_{u>t} u \ .$$

By coarea and isoperimetric ineq, $\alpha(s) = |\{u > s\}|$ satisfies

$$\int_t^\infty \sqrt{\alpha(s)}\, ds \;\leq C \int_t^\infty \alpha(s)\, ds \quad , \quad C = \frac{\lambda}{2\sqrt{\pi}} .$$

Thus $\exists t^* \ : \ \alpha(t) = 0$ for $t \geq t^*$ and $\alpha(t) \geq \frac{1}{C^2}$ for $t < t^*$ □

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Free boundary problem

In view of previous optimality conditions, looking for a special solution amounts to find

- a function $u \in H_0^1(D)$ with

$$\begin{cases} u = const. & \text{in a subset } \Omega \subset D \\ |\nabla u| > 1 & \text{in } D \setminus \Omega \end{cases}$$

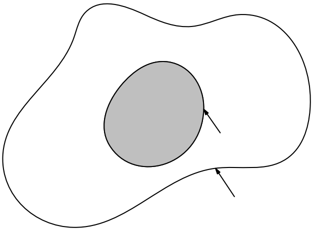
- a vector field $\sigma \in L^2(D; \mathbb{R}^2)$ with

$$\begin{cases} -\mathrm{div}\sigma = \lambda & \text{in } D \\ \sigma = \nabla u & \text{in } D \setminus \Omega \quad (\Rightarrow \; -\triangle u = \lambda \text{ in } D \setminus \Omega) \\ \|\sigma\|_\infty \leq 1 & \text{in } \Omega \end{cases}$$

($\partial\varphi$ satisfies $\partial\varphi(\xi) = \xi$ if $|\xi| > 1$, and $\partial\varphi(0) = \overline{B(0,1)}$.)

Problem in u ?

We are led to a free boundary value problem: find a subset $\Omega = \Omega(u) \subset D$ such that



$$\begin{cases} -\triangle u = \lambda\,, \ |\nabla u| > 1 & \text{in } D \setminus \Omega(u) \\ |\nabla u| = 1 & \text{on } \partial\Omega(u) \\ u \text{ constant on each connected component of} & \Omega(u) \end{cases}$$

BUT needs more in order to construct a σ which fits to λ

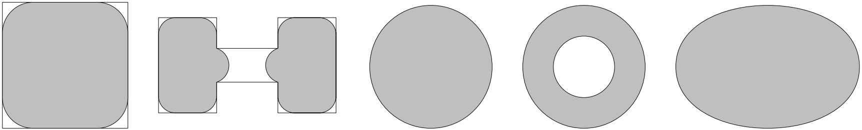
$\Rightarrow$ geometrical condition on set $\Omega(u)$

Let E be a bounded domain of $\mathbb{R}^2$. The *Cheeger constant* of E is defined as

$$h_E := \inf_{\substack{A \subset \overline{E} \\ Per(A) < +\infty}} \frac{|\partial A|}{|A|} = \inf_{\substack{v \in BV_0(E) \\ \int_E v = 1}} \int_E |\nabla v|$$

A minimizer for h_E is called a *Cheeger set* of E. It exists (sub-levels of any v_{opt}), but in general is not unique. However

If E is convex, then: $\exists !$ Cheeger set C_E and $v_{opt} = 1_{C_E}$.



Proposition

The subdifferential of m at the origin is $\partial m(0) = [-h_D, h_D]$. , i.e.

$$\lim_{s\to 0^+} \frac{m(s)}{s} = h_D$$

Remark: the behaviour of $m(s)$ near $s = 0$ is related with the limit $k \to +\infty$ in the original torsion problem.

Proof:

$$m'_+(0) = \lim_{s\to 0^+} \frac{m(s)}{s}$$
$$= \lim_{s\to 0^+} \frac{1}{s} \inf_{\substack{v\in H^1_0(D) \\ \int_D v=1}} \int_D \varphi(s\nabla v) = \inf_{\substack{v\in H^1_0(D) \\ \int_D v=1}} \int_D |\nabla v|$$

where in last line we switch symbols inf and $\int$ $\square$

Calibrable sets

Let $E \subset \mathbb{R}^2$ be a set with finite perimeter. We say that E is *calibrable* if there exists $\sigma \in L^2(E;\mathbb{R}^2)$ (*calibration*) such that

$$\|\sigma\|_\infty \leq 1 \quad , \quad -\mathrm{div}\sigma = h_E \quad , \quad [\sigma \cdot \nu_E] = -1 \quad \mathcal{H}^1 - \text{a.e. on } \partial E$$

Proposition

Let $E \subset \mathbb{R}^2$ be a bounded domain with finite perimeter. Then

$$E \text{ calibrable} \iff E \text{ is Cheeger set of itself}$$

The proof follows from divergence Theorem and the fact that:

$$h_E = \max\{\lambda \in \mathbb{R} \ : \ \exists \sigma \in L^2(E;\mathbb{R}^2)\,,\ \|\sigma\|_\infty \leq 1\,,\ -\mathrm{div}\sigma = \lambda\}\ .$$

Remark: If E is convex, then E calibrable $\iff \|H_{\partial E}\|_\infty \leq \frac{|\partial E|}{|E|}$

Revisited free boundary problem

Looking for special solutions amounts to find a "plateau" $\Omega \subset D$ (smooth enough) so that

- Ω is **calibrable**
- There exits a solution $u \in H^1_0(D)$ to the **overdetermined** problem

$$\begin{cases} -\triangle u = h_\Omega \quad , \quad |\nabla u| > 1 & \text{in } D \setminus \Omega \\ |\nabla u| = 1 & \text{on } \partial\Omega \\ u \text{ constant on each connected part of} & \partial\Omega \end{cases}$$

Vanishing volume fraction (limit plateau as $s \to 0$)

$$\lim_{k\to+\infty} \frac{\phi(k)}{\sqrt{2k}} = \inf\left\{ C^{\lim}(\mu) + \frac{1}{2}\int d\mu \ : \ \mu \in \mathcal{M}^+(Q) \right\}$$
$$= \min_{\sigma\in\mathcal{M}(Q;\mathbb{R}^2)} \left\{ \int |\sigma| \ : \ \mathrm{div}_{x'}\sigma = 0\,,\ \int_D (x_1 d\sigma_2 - x_2 d\sigma_1) = \gamma(x_3) \right\}$$

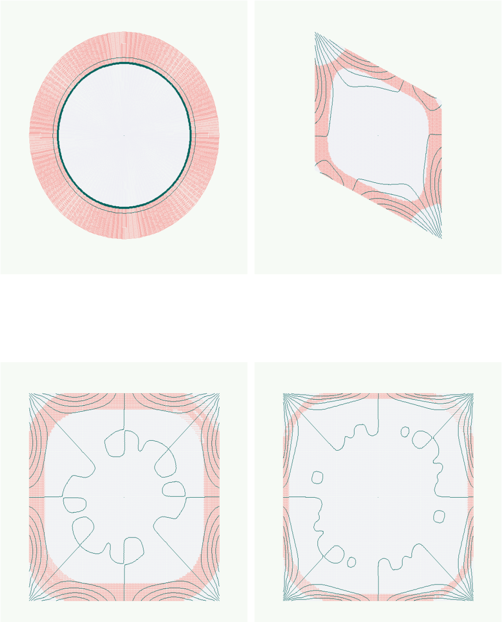
Then ν optimal $\iff$ $\nu = (-\partial_2 u, \partial_1 u)$, with u optimal for

$$\min\left\{ \int |Du| \ : \ u \in BV(\mathbb{R}^2)\ ,\ u = 0 \text{ in } \mathbb{R}^2 \setminus \overline{D}\ ,\ \int_D u = 1 \right\} = h_D$$

Thus if D is convex, the optimal stress concentrates on ∂C_D:

Material concentrates on the boundary of the Cheeger set of D

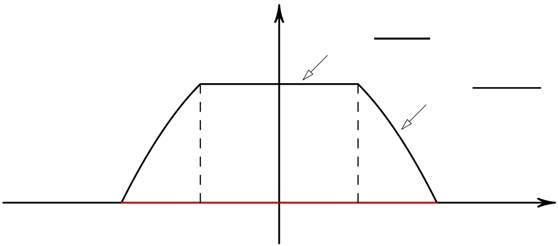
Some numerical computations



Proposition (radial case)

Let D be the ball $B(0, R)$. For every $s \in \mathbb{R}$ there exists a special solution u for $m(s)$.

Proof: If $s > 0$, let u be defined as follows:



where $r \in (0, R)$ is the unique solution of $s = \frac{\pi}{4r}(R^4 - r^4)$.

Here $\Omega(u) = \{|x| < r\}$. The dual solution $\sigma = -\frac{x}{r}$ satisifies $|\sigma| \leq 1$ on $\Omega(u)$. □

Is the disk the unique domain ?

Recall: existence of a special solution is equivalent to existence of optimal shape.

- The answer is *yes* (among C^1 domains) for a similar variational problem, corresponding to maximizing the torsional rigidity of rods with a given cross-section D by mixing two linearly elastic materials in fixed proportions. [Murat, Tartar]
- But.... the answer is *no* (even among analytic domains) for our problem!

Reason why: Our integrand $\varphi(z)$ is *not differentiable at* $z = 0$ (it would be C^1 if the void is replaced by a weak material)

Special solutions for D not a ball

Theorem

There exists a domain D (different from a ball) and a parameter $s \in \mathbb{R} \setminus \{0\}$ such that $m(s)$ admits a special solution u.
Moreover D and the plateau $\Omega(u)$ is convex with analytic boundary.

Sketch of proof: We need to construct a bounded analytic domain D such that there exist

- a function $u \in H_0^1(D)$ with

$$\begin{cases} \nabla u = 0 \quad \text{in a convex set} \quad \Omega \subset D \\ |\nabla u| > 1 \quad \text{in } D \setminus \Omega \\ \displaystyle\int_D u = s\,, \ \text{for some } s \in \mathbb{R} \setminus \{0\}\,, \end{cases} \tag{1}$$

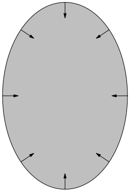
- a field $\sigma \in L^2(D;\mathbb{R}^2)$ with

$$\begin{cases} |\sigma| \leq 1 \quad \text{in } \Omega\,, \\ \sigma = \nabla u \quad \text{in } D \setminus \Omega\,, \\ -\text{div}\sigma = \lambda \quad \text{in } D\,, \ \text{for some } \lambda \in \mathbb{R}\,. \end{cases} \tag{2}$$

- **Step1**

We consider Ω bounded, convex, with analytic boundary, and such that $\|H_{\partial\Omega}\| < |\partial\Omega|/|\Omega|$.

(known fact) $\Rightarrow$ Ω is Cheeger set of itself, i.e. it is calibrable.



Let $\sigma_1 \in L^2(\Omega;\mathbb{R}^2)$ be a calibration for Ω, then

$$\begin{cases} \|\sigma_1\|_\infty \leq 1 & \text{in } \Omega \\ -\text{div}\sigma_1 = h_\Omega & \text{in } \Omega \\ [\sigma_1 \cdot \nu_\Omega] = -1 & \mathcal{H}^1 - \text{a.e. on } \partial\Omega \end{cases}$$

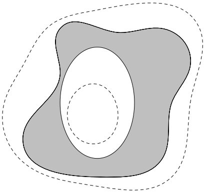
• **Step2**

$\partial\Omega$ analytic $\overset{Cauchy-Kowalevskaya}{\Longrightarrow}$ $\exists$ v analytic solution of

$$\begin{cases} -\triangle v = h_\Omega & \text{in } \mathcal{N} \\ v = 1\,,\ -v_\nu = 1 & \text{on } \partial\Omega \end{cases}$$

in a neighbourhood $\mathcal{N}$ of $\partial\Omega$.

Moreover there exists a curve $\gamma \subset \mathcal{N}$ analytic that is the boundary of some domain $D \supset \Omega$, such that



$$\begin{cases} -\triangle v = h_\Omega & \text{in } D \setminus \Omega \\ |\nabla v| > 1 & \text{in } D \setminus \Omega \\ v = 1\,,\ v_\nu = -1 & \text{on } \partial\Omega \\ v = 1 - \varepsilon & \text{on } \partial D \end{cases}$$

for some $0 < \varepsilon < 1$.

- **Step3**

The functions

$$u(x) := \begin{cases} \varepsilon & \text{in } \Omega \\ v - (1-\varepsilon) & \text{in } D \setminus \Omega \end{cases}, \quad \sigma(x) := \begin{cases} \sigma_1 & \text{in } \Omega \\ \nabla v & \text{in } D \setminus \Omega \end{cases}$$

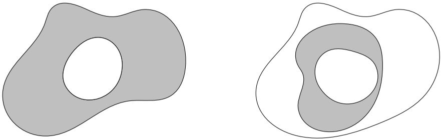
satisfy the conditions (1) and (2).
In addition Ω is convex and D, Ω have analytic boundary. □

5. Further properties of special solutions

- **Property 1:** If the value function $m(s)$ is affine on some $[\alpha, \beta]$, then no special solution exists for $\alpha < s < \beta$.
- **Property 2:** If $m(s)$ is *stricly convex* on $[\alpha, \beta]$, then there exists a unique solution for $\alpha < s < \beta$.
- **Property 3:** Let D be convex and assume that u is a special solution with smooth connected $\Omega(u)$ such that $\Omega(u) \subset\subset D$. Then $\Omega(u)$ is convex.
  (proof uses P-functions and Hopf's Lemma)

- **Property 4:** Assume that D is *not* Cheeger set of itself, and let $s_\varepsilon \searrow 0$. Then problem $m(s_\varepsilon)$ cannot admit for every ε a special solution u_ε with $\Omega(u_\varepsilon) \subset\subset D$.

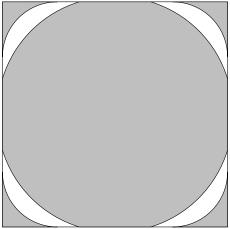
- **Property 5:** Assume that u is a special solution with smooth $\Omega(u)$. Then each connected component of $D \setminus \Omega(u)$ meets the boundary ∂D.
  So cannot have with ($\Omega(u)$ in dark)



Open problems

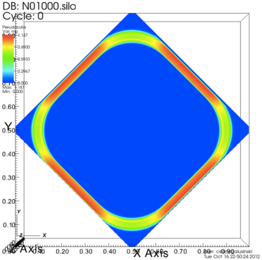
- Regularity of the free boundary
  [Caffarelli, Petrosyan, Salazar, Shahgholian]

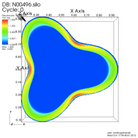
- Non-existence of special solutions ? *e.g.* in case of the square
  Kawohl, Stara, Wittum and more recently C. Galusinski, E. Oudet



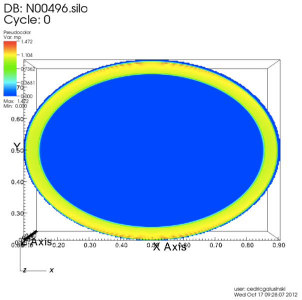
A possible plateau for a special solution on the square.

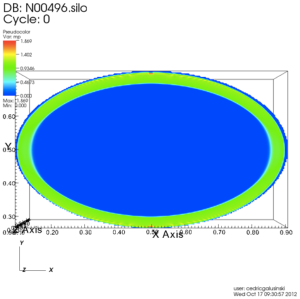
Some numerics by C. Galusinski (IMATH-Toulon)





red or yellow zone $\theta = 1$, blue: zones $\theta = 0$





green: zones with homogenization , blue: zones $u = cte$

Conjectures

- There exits a special solution if D is a convex C^2 subset such that $\|H_{\partial D}\|_\infty \leq \frac{|\partial D|}{|D|} = h_D$.
- For a larger class of domains (including convex domains), there exists s^* such that:
  a special solution exists for $m(s)$ for all $s \geq s^*$

References

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- Lucardesi, Concentration phenomena in the optimal design of thin rods *Journal of Convex analysis, 2014.*

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